MATC44 Week 8 Notes

1. Bijection:

- Two infinite sets, A and B, have the same cardinality (number of elements) iff it is possible to match each element of A to an element of B s.t. every element of each set has exactly one "partner" in the other set.

I.e. $|A| = |B| = \infty$ iff there exists a bijective function f where $f: A \to B$

- There are bijections between the following sets:
  a) Natural numbers and integers
  b) Natural numbers and rational nums

- Note: $|A| = |B| = \infty$ does not imply that there is a bijective function f s.t. $f: A \to B$. E.g. There is no bijection between the natural numbers and real numbers.

2. Permutations:

- Consider n objects, for example, consider the first n natural numbers. $T_n = \{1, 2, \ldots, n\}$. A permutation of T_n is a rearrangement of T_n where it is ordered and every element appears exactly once.

I.e. It is a new ordered list where all the natural numbers from 1 to n appear exactly once, in some specific order.
- E.g. Let $T_3 = \{1,2,3\}$. The following are the permutations of T_3: $(1,2,3)$, $(1,3,2)$, $(2,1,3)$, $(2,3,1)$, $(3,1,2)$, $(3,2,1)$.
- In general, there are $n!$ permutations of T_n and hence, $n!$ permutations of any n distinct objects.
- Any permutation can be determined by first determining its first entry (there are n possible ways for this), then determining its second entry (there are $n-1$ ways for this) until we reach the final entry (there's 1 way for this). By the multiplicative principle the entire process can be done in $n(n-1)(n-2)\dots(1)$ or $n!$ ways.

3. Ordered Subsets:
- An ordered subset with k elements of T_n is a list which consists of k elements taken from T_n s.t. no repetitions are allowed and the list is ordered.
- There are $\frac{n(n-1)(n-2)\dots(n-(n-1))}{(n-k)\dots(n-(n-1))}$ or $\frac{n!}{(n-k)!}$ ordered subsets of T_n with k elements.

- Given n objects, there are O_n^2 ordered subsets of pairs of numbers.

$$O_n^k = \frac{n!}{(n-k)!}, \quad \text{so} \quad O_n^2 = \frac{n!}{(n-2)!}$$

I.e. There are $\frac{n!}{(n-2)!}$ pairs given n numbers.

E.g. $n=2$, there's 2 pairs we can make.

$$\frac{2!}{(2-2)!} = \frac{2!}{0!} = \frac{2!}{1} = 2! = 2$$

E.g. $n=3$, there's 6 pairs we can make.

$$\frac{3!}{(3-2)!} = \frac{3!}{1!} = 3! = 6$$

- Note: If $k=n$, then we get $\frac{n!}{(n-k)!}$ which is equal to $\frac{n!}{(n-n)!}$ or $n!$.

This means that if $k=n$, we have a permutation of the n objects.

4. Combinations:

- A combination of T_n with k elements is a subset of T_n with k elements for which repetition is not allowed and order does not matter.
I.e. The subset is unordered.

- Note that the only difference between combinations with k elements and ordered subsets of k elements is that order does not play a role in combinations whereas order plays a role in ordered subsets. Since there are $\frac{n!}{(n-k)!}$ ordered subsets with k elements and there are $k!$ different re-orderings (permutations) of any ordered subset with k elements, there are $\frac{n!}{(n-k)!\,k!}$ different combinations with k elements.

- Note: Let $T_3 = \{1,2,3\}$. Then, we have the following ordered subsets with 3 elements: $\{1,2,3\}$, $\{2,1,3\}$, $\{1,3,2\}$, $\{2,3,1\}$, $\{3,1,2\}$, $\{3,2,1\}$. However, we only have 1 combination with 3 elements. This is because since order doesn't matter, $\{1,2,3\} = \{1,3,2\} = \{2,1,3\} = \{2,3,1\} = \{3,1,2\} = \{3,2,1\}$

- A combination of n objects with k elements is also known as "n choose k" and can be denoted by $\binom{n}{k}$.

– The number $\binom{n}{k}$ is called the binomial coefficient because of the following property:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

$$= \binom{n}{n} x^n + \ldots + \binom{n}{2} x^2 + \binom{n}{1} x + \binom{n}{0}$$

This identity follows easily by observing the following: For each k, the term x^k is formed by choosing k of the n factors of the product $(1+x)^n = (1+x)\cdot(1+x)\ldots\cdot(1+x)$. Hence, there are $\binom{n}{k}$ ways to choose k factors and as such, there are $\binom{n}{k}$ terms of the form x^k in the expanded sum.

– Here are some identities of the binomial coefficients:

a) Special Cases: $\binom{n}{0} = 1$, $\binom{n}{1} = n$, $\binom{n}{2} = \frac{n(n-1)}{2}$,

$$\binom{n}{n} = 1, \quad \binom{n}{n-1} = n$$

b) Symmetry Identity: $\binom{n}{k} = \binom{n}{n-k}$

c) Pascal Identity: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

d) $\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n-1} + \binom{n}{n} = 2^n$

e) $0\cdot\binom{n}{0} + 1\binom{n}{1} + 2\binom{n}{2} + \ldots + n\binom{n}{n} = n\cdot 2^{n-1}$

f) $\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$

Most of the above identities can be proven algebrically, but all of them can be proven combinatorically.

Proof of Symmetry Identity:

a) Algebraic Proof:

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

$$\binom{n}{n-k} = \frac{n!}{(n-k)!\,(n-(n-k))!} = \frac{n!}{(n-k)!\,k!} = \binom{n}{k}$$

b) Combinatorial Proof:

- Consider the bijection principle.
- There's a bijection between combinations with k elements and elements with n-k elements because if we choose a combination with k elements, then we have left out n-k elements which form an n-k combination.

- Hence, by the bijection principle, the number $\binom{n}{k}$ of all combinations with k elements are equal to the number $\binom{n}{n-k}$ of all combinations with n-k elements.

Remark: Recall that a function is bijective if it is both one-to-one and surjective. This means that each distinct element of the domain is mapped to exactly one distinct element of the co-domain. Furthermore, no elements in the co-domain is unmapped or mapped multiple times. This means that the cardinality of the domain and co-domain must be the same.

E.g. Suppose that the function f is bijective and that there are 3 elements in its domain. Since we know each of these 3 elements maps to exactly 1 distinct element in the co-domain and that no element in the co-domain is not mapped to or mapped to multiple times, then there must be 3 elements in the co-domain.

Proof of Pascal Identity:

a) Algebraic Proof:

$$\binom{n}{k} = \frac{n!}{(n-k)!\,(k)!}$$

$$\binom{n-1}{k} = \frac{(n-1)!}{(k)!\,(n-1-k)!}$$

$$\binom{n-1}{k-1} = \frac{(n-1)!}{(k-1)!\,(n-1-(k-1))!}$$

$$= \frac{(n-1)!}{(k-1)!\,(n-k)!}$$

$$\binom{n-1}{k} + \binom{n-1}{k-1}$$

$$= \frac{(n-1)!}{(k)!\,(n-k-1)!} + \frac{(n-1)!}{(k-1)!\,(n-k)!}$$

$$= \frac{(n-1)!}{\underline{(k)}(k-1)!\,(n-k-1)!} + \frac{(n-1)!}{\underline{(n-k)}(k-1)!\,(n-k-1)!}$$

$$= \frac{(n-1)!\,(n-k) + (n-1)!\,(k)}{(n-k)!\,(k)!}$$

$$= \frac{(n-1)!\,[n-k+k]}{(n-k)!\,k!}$$

$$= \frac{n!}{(n-k)!\,k!} = \binom{n}{k}$$

b) Combinatorial Proof:

- Consider the set T_n where $T_n = \{1, 2, \ldots, n\}$, and consider the element n. If we have a combination of k elements from these n objects, then the element n is either part of the k elements or not part of the k elements.

If the element n is part of the k elements, then we have $k-1$ elements left to choose from $n-1$ objects, so we get $\binom{n-1}{k-1}$.

If the element n is not part of the k elements, then we have k elements to choose from $n-1$ objects (It's $n-1$ because we know that n will not be chosen.) Therefore, we have $\binom{n-1}{k}$.

By the additive principle, we get:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Proof of Identity d)

I.e. $\binom{n}{0}+\binom{n}{1}+\ldots+\binom{n}{n}=2^n$

a) Algebraic Proof:

Recall that $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k}\cdot x^k$.

If we set $x=1$, we get:

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

b) Combinatorial Proof:

Consider the following:

i) The left hand side (LHS) counts all the possible combinations that we can form.

ii) The right hand side (RHS) counts the number of ways to go through each of the n objects and either choose it or not. Since there are n elements, each with 2 possibilities, there are 2^n possible combinations we can form.

Proof of Identity e)

I.e. $0 \cdot \binom{n}{0} + 1 \cdot \binom{n}{1} + \ldots + n\binom{n}{n} = n \cdot 2^{n-1}$

a) Algebraic Proof:

Recall that $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$

Differentiating the above equation and setting $x=1$ gives us:

$$n(1+x)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} \cdot k \cdot x^{k-1}$$

$$n \cdot 2^{n-1} = \sum_{k=0}^{n} \binom{n}{k} \cdot k$$

b) Combinatorial Proof:

Consider the following:

The term $k\binom{n}{k}$ counts all possible teams with k players with an assigned leader. I.e. Given k players, we can create $\binom{n}{k}$ teams, and since the assigned leader is one of the k players, there are k ways to choose the leader.

E.g. Suppose we want 5 players on the team and we want 1 of them to be the leader. Let these 5 players be A, B, C, D, and E. There are $\binom{n}{5}$ ways to create a team of 5 players. Furthermore, there are 5 ways we can select a leader. Either A will be the leader or B or C or D or E. Therefore, there are $5 \cdot \binom{n}{5}$ ways to select a team of 5 with an assigned leader.

i) The LHS counts the number of all teams with an assigned leader.

ii) The RHS first the leader first. Since there are n players, there are n ways to choose a leader. For each of the remaining $(n-1)$ players, we have 2 possibilities: either they are on the team or not. Hence, by the multiplicative principle, we have $n \cdot 2^{n-1}$ different ways to make a team with a leader.

Proof of Identity f)

I.e. $\binom{n}{0}^2 + \binom{n}{1}^2 + \dots + \binom{n}{n}^2 = \binom{2n}{n}$

Note: There is no algebraic proof for this, just a combinatorial proof.

Combinatorial Proof:

Consider the symmetry identity. From it, we know that $\binom{n}{k} = \binom{n}{n-k}$. Therefore, we can rewrite $\binom{n}{0}^2 + \binom{n}{1}^2 + \dots + \binom{n}{n}^2$ as

$$\sum_{k=0}^{n} \binom{n}{k} \cdot \binom{n}{n-k}.$$

$$\sum_{k=0}^{n} \binom{n}{k} \cdot \binom{n}{n-k} = \binom{2n}{n}$$

The RHS counts all combinations with n elements from $2n$ objects. We can divide these $2n$ objects into 2 sets S_1 and S_2 s.t. S_1 and S_2 have n objects each. Hence, the RHS counts all combinations with n elements that can be formed from using the elements in S_1 and S_2.

Consider the LHS now. Any n combination of the 2n elements of the set $S_1 \cup S_2$ will contain k elements from S_1 and n-k elements from S_2, for some k between 0 and n. There are $\binom{n}{k}$ ways to choose k elements from S_1 and $\binom{n}{n-k}$ ways to choose the n-k elements from S_2. By the multiplicative principle, there are $\binom{n}{k} \cdot \binom{n}{n-k}$ such combinations. The result follows from the additive principle where we sum all possibilities for k.

- There are 3 general methods to prove identities in a combinatorial way:

a) The first method makes use of the bijection principle. That is, we need to find a bijection between the elements counted on the LHS of the identity and the elements counted on the RHS of the identity. The proof for the symmetry identity uses this method.

b) The splitting method. Recall that for the combinatorial proof of Pascal's Identity, we split the set of all combinations into 2 groups, all combinations with n and all combinations without n.

c) Double counting. This method computes the elements of the same set in 2 different ways. Since the same number is computed in 2 different ways, the final results from these 2 ways must coincide leading to the desired identity.

5. Combinations with Repetition:

- A combination with repetition with k elements of n objects is an unordered collection of k elements from n given objects where each element might appear repeatedly up to k times.

- Denoted as $|E_n^k|$ or $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$.

- E.g. Consider the set $T_3 = \{1,2,3\}$. There are 3 combinations of 2 elements from the above 3 objects, namely $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$.

  But, there are 6 combinations with repetition of 2 elements from the above 3 objects, namely $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{1,1\}$, $\{2,2\}$, $\{2,3\}$.

- Thm: $\begin{bmatrix} n \\ k \end{bmatrix} = \binom{n-1+k}{k}$

Proof:

We observe that any combination with repetition with k elements from the set $\{1, 2, \ldots, n\}$ gives rise to a seq of $(n-1)+k$ objects s.t. k of them are the k objects of the combination with repetition and the $(n-1)$ represents the arrows used to declare that we are done with using one number in our combination and we are moving to the next number.

E.g. Consider the set $\{1,2,3\}$ and suppose we want a combination with repetition with 2 elements.

a) Let the 2 elements be 1 and 2.

$$n-1+k = 3-1+2$$
$$= 4$$

$\therefore$ We have 4 entries, 2 of which are arrows and the other 2 are the nums 1 and 2.

The entries look like this:
$(1, \rightarrow, 2, \rightarrow)$

The above list provides a way to go through the set $\{1,2,3\}$ in the following way:

i) We write 1 as the first entry since 1 is a member of the combination.

ii) We are done with 1, so we use an arrow, $\rightarrow$, to declare that we need to move to 2.

iii) We write 2 as the next entry since 2 is a member of the combination.

iv) We are done with 2, so we use an arrow, $\rightarrow$, to declare that we need to move to 3. But, since 3 is not a member of the combination, the process terminates.

b) Let the 2 elements be 2 and 2.

$$n-1+k = 3-1+2$$
$$= 4$$

The entries look like this:

$(\rightarrow, 2, 2, \rightarrow)$

The above provides a way to go through the set $\{1, 2, 3\}$ in the following way:

i) 1 is not a member of the combination, so we use an arrow to move to 2.

ii) We write 2 as the next entry since it's a member of the combination.

iii) We write 2 as the next entry since it's a member of the combination.

iv) We are done with 2 so we use an arrow to move to 3. However, 3 isn't a member of the combination, so the process terminates.

More generally, every combination with repetition with k elements produces a unique seq with $n-1+k$ entries ($n-1$ arrows and k objects). Every such seq is uniquely determined by the location of the $n-1$ arrows. Hence, it suffices to know in how many ways we can place these $n-1$ arrows in $n-1$ of the $n-1+k$ entries of the seq. This can be done in $\binom{n-1+k}{n-1}$ ways.

However, recall the symmetry identity: $\binom{n}{k} = \binom{n}{n-k}$.

$$\binom{n-1+k}{n-1} = \binom{n-1+k}{n-1+k-(n-1)}$$
$$= \binom{n-1+k}{k}$$